Negative Exponents and Ratios in Wallis

We have often found it interesting to examine some of the ideas in mathematics that did not gain general acceptance. The serious consideration of these alternative conceptions can enlighten our thinking and our teaching practice as we try to understand student conceptions. The following examination of Wallis' use of negative values within his theory of index and ratio is a good example.

Wallis interpreted negative numbers as exponents in the same way that we do. That is, he defined the index of $1/x$ as -1, and the index of $1/x^2$ as -2, and so on. He also extended this definition to fractions; for example, $1/\sqrt{x}$ has an index of -1/2. He then claimed that the relationship between the index and the characteristic ratio is still valid for these negative indices. That is, that if k is the index then 1/(k+1) is the ratio of the area under the curve (shaded) to the rectangle (see Figure 5a). In the case of a negative index this shaded area is unbounded. This did not deter Wallis from generalizing his claim.

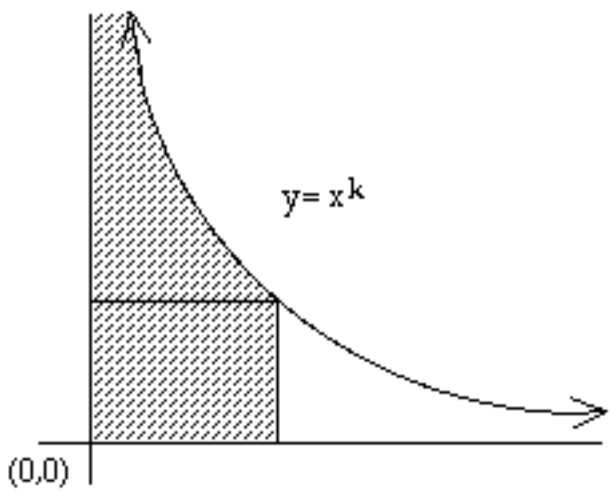

Fig. 5a

When $k = -\frac{1}{2}$, the characteristic ratio should be $\dfrac{1}{-\frac{1}{2}+1} = 2$. This value is indeed correct, for the unbounded area under the curve $y = 1/\sqrt{x}$ does converge to twice the area of the rectangle. This is true no matter what right hand endpoint is chosen.

When $k = -1$, the characteristic ratio should be $\dfrac{1}{-1+1} = \dfrac{1}{0} = \infty$ (Wallis introduced this symbol for infinity into mathematics). Wallis accepted this ratio as reasonable since the area under the curve y = 1/x diverges. This can be seen from the divergence of the harmonic series

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots=\infty$, which had been known since at least the fourteenth century (Boyer, 1968, Chap. XIV).

When $k=-2$, the characteristic ratio should be $\frac{1}{-2+1}=\frac{1}{-1}$. Here Wallis' conception of ratio differs from our modern arithmetic of negative numbers. He did not believe that 1/-1 = -1. Instead he stayed with his epistemology of multiple representations. Since the shaded area under the curve $y=1/x^2$ is greater than the area under the curve y =1/x, he concluded that the ratio 1/-1 is greater than infinity ("ratio plusquam infinita") (Nunn, 1909-1911, p. 355). He went on to conclude that 1/-2 is even greater. This explains the plural in the title of his treatise *Arithmetica Infinitorum*. The appropriate translation would be *The Arithmetic of* *Infinities*.

Most historians of mathematics quickly brush over this concept if they mention it at all. Those who mention it quickly cite the comments of the French mathematician Varignon (1654 - 1722), who pointed out that if the minus sign is dropped in the ratio then we arrive at the correct ratio of the unshaded area under the curve to the area of the rectangle. This was an instance of the beginning of the idea that negative numbers could be viewed as complements or reversals of direction.

We, however, find it well worth pondering Wallis' original conception. In what ways does it make sense to consider the ratio of a positive to a negative number as greater than infinity? In the area interpretation from Figure 5a, we could view these different infinities as greater and greater rates of divergence. Such views are often taken in mathematics. The area under $y=1/x^3$ does diverge faster than the area under $y=1/x^2$.

Let's consider an even simpler situation. If I have $1, and you have 50¢, then we say that I have twice as much money as you. If I have $1, and you have 10¢ then we say that I have ten times as much money as you. If I have $1, and you have nothing, then we could say that I have infinitely more money than you. Many mathematicians would accept this statement. Now if I have $1, and you are in debt; shouldn't we say that the ratio of my money to yours is even greater than infinity? This is a question that is worth pondering.

References cited in the text can be found at
http://www.quadrivium.info/MathInt/Notes/WallisNewtonRefs.pdf